

# Spectral singularities in $\mathcal{PT}$ -symmetric Bose-Einstein condensates

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**Abstract.** We consider the model of a  $\mathcal{PT}$ -symmetric Bose-Einstein condensate in a delta-functions double-well potential. We demonstrate that analytic continuation of the primarily non-analytic term  $|\psi|^2\psi$  – occurring in the underlying Gross-Pitaevskii equation – yields new branch points where three levels coalesce. We show numerically that the new branch points exhibit the behaviour of exceptional points of second and third order. A matrix model which confirms the numerical findings in analytic terms is given.

PACS numbers: 03.65.Ge, 03.75.Hh, 11.30.Er, 31.15.-p, 02.30.-f

Submitted to: *J. Phys. A: Math. Gen.*

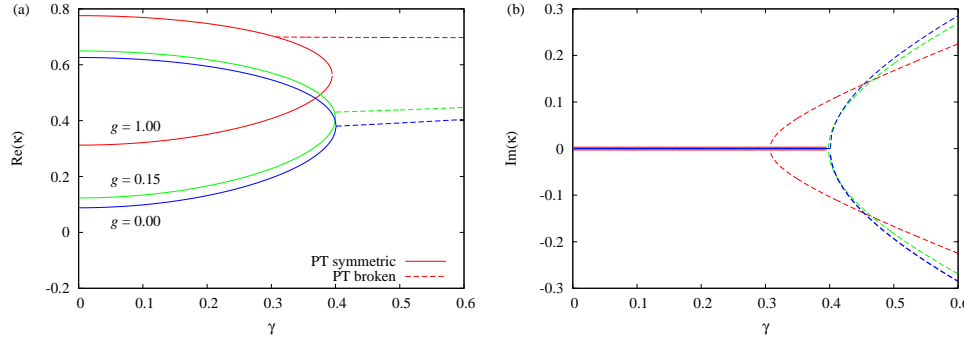
## 1. Introduction

It is a characteristic feature of non-Hermitian but  $\mathcal{PT}$ -symmetric Hamiltonians that, as an external parameter is varied, pairs of real eigenvalues coalesce at an exceptional point (EP) and turn into complex conjugate pairs (Bender and Boettcher [1], Moiseyev [2]). Recent investigations of *non-linear*  $\mathcal{PT}$ -symmetric Hamiltonians modelling Bose-Einstein condensates in double-well potentials with sink and source terms of atoms at the respective wells have revealed a specific spectral behaviour which implies apparently new spectral singularities [3–9]. Pairs of real eigenvalues merge at an EP when the strength of the non-Hermiticity is increased, but no pairs of complex conjugate eigenvalues appear beyond the EP. Rather, such pairs are born as if “out of nowhere” *before* the EP, that is for smaller values of the loss/gain term, where they bifurcate from the branch of the ground state eigenvalue.

The reason for this unusual behaviour is the non-analyticity of the non-linear term  $|\psi|^2\psi$  in the underlying Gross-Pitaevskii equation. Numerical calculations for Bose-Einstein condensates in  $\mathcal{PT}$ -symmetric double-well potentials have shown [7–9] that by appropriate analytic continuation of the non-linear term pairs of complex conjugate eigenstates indeed arise at the branch point. Moreover, in an analytically solvable toy model for a  $\mathcal{PT}$ -symmetric two-mode Bose-Einstein condensate Graefe [6] used a particular analytic extension of eigenvalues and eigenstates. This way it has been demonstrated that the “prematurely” born pairs of complex conjugate eigenvalues emerge from an additional pair of real eigenvalues. Therefore three real eigenvalues effectively coalesce at what we shall term a triple point.

The features of a non-linear term of the form  $|\psi|^2\psi$  have become very relevant for the study of  $\mathcal{PT}$ -symmetric physical systems since this type of non-linearity not only appears in the mean-field description of Bose-Einstein condensates.  $\mathcal{PT}$ -symmetric optical setups of coupled dual waveguides including a Kerr non-linearity show exactly the same structure in the underlying equations. They have been used to demonstrate the existence of uni-directional structures [10] or the presence of solitons in loss/gain media [11–16]. The remarkable success of realising  $\mathcal{PT}$  symmetry and  $\mathcal{PT}$ -symmetry breaking experimentally emphasises the relevance of these optical setups [17, 18]. The additional properties of the combination with a non-linearity as the possibility of uni-directionality [10] might lead to new technical devices. In quantum mechanics non-linear  $\mathcal{PT}$ -symmetric systems have been discussed in model potentials by Musslimani et al. [19], and for Bose-Einstein condensates described in a two-mode approximation [3–6] or by solving the mean-field limit of the Gross-Pitaevskii equation in position space [7–9].

Potentials consisting only of delta-functions often provide a very simple access to solutions, in many cases they can be obtained analytically. For example, the spontaneous symmetry breaking in double-well structures has been investigated analytically in the limit of infinitely narrow potential wells described by delta-functions by Maytevarunyoo et al. [20]. Rapedius and Korsch investigated the decay of Bose-Einstein condensates in a double-delta-setup [21]. Bifurcations of stationary solutions have been studied by Witthaut et al. [22] for Bose-Einstein condensates in a delta-comb. With the exception of the study of Bose-Einstein condensates in a  $\mathcal{PT}$ -symmetric double-delta-trap [7, 8], the discussion of  $\mathcal{PT}$ -symmetry in delta-potentials has been restricted to linear quantum mechanics. Jakubský and Znojil [23] considered analytical solutions of a particle in an infinitely high square well containing two purely imaginary delta-functions in a  $\mathcal{PT}$ -symmetric arrangement. The influence of a pair of



**Figure 1.** Real parts (a) and imaginary parts (b) of the eigenvalues  $\kappa$  of the full non-linear equation (1) as functions of the loss/gain parameter  $\gamma$  for  $a = 2.2$  for the three values of the non-linearity parameter (in (a) from bottom to top)  $g = 0.0, 0.15$  and  $1.0$ . Solid lines denote purely real eigenvalues, dashed lines complex conjugate eigenvalues. For  $g \neq 0$  the complex conjugate eigenvalues bifurcate from the ground state branch before the branch point where the two real solutions coalesce.

$\mathcal{PT}$ -symmetric delta-functions on the bound and scattering states of a single real delta-function has been investigated by Jones [24]. The spectral properties, in particular bound states and spectral singularities, in non-Hermitian delta-potentials have been studied by Mostafazadeh et al. [25–27].

It is the purpose of the present paper to study the new types of spectral singularities in more detail. We do this in Section 2 by solving numerically an analytically continued version of the Gross-Pitaevskii equation for the model of a Bose-Einstein condensate in a  $\mathcal{PT}$ -symmetric double delta potential. We find that the triple point can exhibit the behaviour of a second-order (EP2) or a third-order (EP3) exceptional point depending on the particular encirclement. In a matrix model presented in Section 3 this behaviour is described in analytic terms.

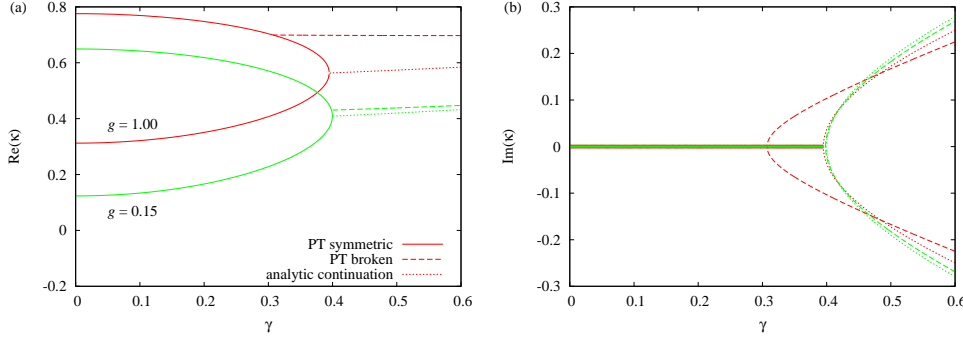
## 2. Complexification of the Gross-Pitaevskii equation

### 2.1. Model with delta-function traps

The model of a Bose-Einstein condensate trapped in two  $\mathcal{PT}$ -symmetric potential wells represented by delta-functions has been introduced in [7, 8]. Here we only recall the essential points which are relevant to the present discussion. In the mean-field description the non-linear Gross-Pitaevskii equation to be solved reads in dimensionless units

$$\begin{aligned}
 -\psi''(x) - [(1 + i\gamma)\delta(x + a/2) + (1 - i\gamma)\delta(x - a/2)]\psi(x) \\
 - g|\psi(x)|^2\psi(x) = -\kappa^2\psi(x).
 \end{aligned}
 \tag{1}$$

Here  $g$  is the non-linear interaction strength, the real-valued parameter  $\gamma$  determines the strength of the gain and loss terms in the two delta wells located at  $\pm a/2$ . (The relation to physical quantities can be found in [7, 8] as well as details on the numerics.) Since we are interested in the complete eigenvalue spectrum, including complex eigenvalues, solutions are searched with  $\kappa \in \mathbb{C}$ . Figure 1 recapitulates the results for the eigenvalues  $\kappa$  for different values of the non-linearity parameter  $g$ . It can



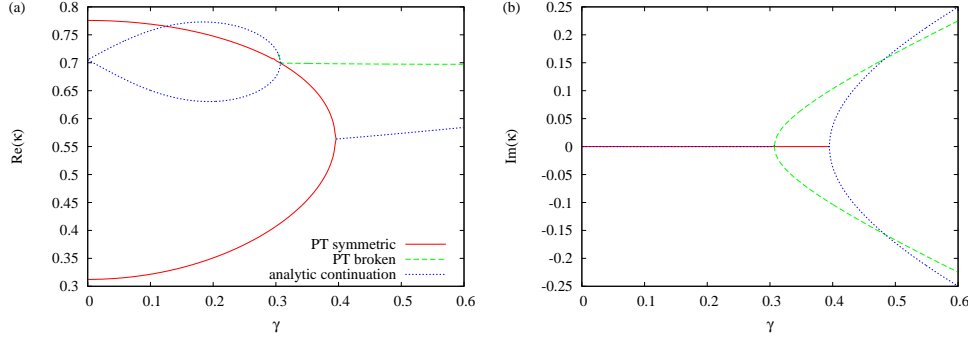
**Figure 2.** Real parts (a) and imaginary parts (b) of the eigenvalues  $\kappa$  for the states appearing at  $g = 1$  and  $g = 0.15$ . Using the substitution  $|\psi(x)|^2 \rightarrow \psi(x)\psi(-x)$  the analytic continuation beyond the branch point  $\gamma_{\text{bp}}$  of the two real  $\mathcal{PT}$ -symmetric eigenvalue states is obtained.

be seen that for every value of  $g$  a pair of real eigenvalues exists up to some value  $\gamma_{\text{bp}}$ , where they coalesce at a branch point. Pairs of complex conjugate eigenvalues emerge at critical values  $\gamma_{\text{cr}} < \gamma_{\text{bp}}$ , where  $\gamma_{\text{cr}}$  decreases for increasing  $g$ . For sufficiently large non-linearity these solutions already appear for  $\gamma = 0$ . Thus one has ranges of  $\gamma$  where two real and two complex conjugate eigenvalues coexist. The wave functions show that those belonging to the real eigenvalues are themselves  $\mathcal{PT}$ -symmetric, i.e. they are eigenfunctions of  $\mathcal{PT}$ , whereas the two states with complex eigenvalues  $\kappa$  are not.

Figure 1 reveals that the branching-off of the real eigenvalue changes continuously from the branch for  $g = 0$ . The wave functions, however, show a non-uniform limit as  $g$  tends to zero: for every value  $g > 0$  their asymptotic form is given by  $1/\cosh(\kappa x)$ , while at  $g = 0$  it is  $\exp(\pm \kappa x)$ . The non-uniform behaviour of  $g \rightarrow 0$  will be addressed in a few cases below.

Pairs of complex conjugate eigenvalues emerge at the branch points only when the non-analytic term  $|\psi(x)|^2$  in (1) is continued beyond the branch points. This is illustrated in figure 2. For the analytic continuation we exploit the  $\mathcal{PT}$ -symmetry of the wave functions corresponding to the real eigenvalues, i.e.  $\psi^*(x) = \psi(-x)$ . Therefore, when approaching the branch point we replace the non-linear term  $g|\psi(x)|^2$  for the  $\mathcal{PT}$ -symmetric states by  $g\psi(x)\psi(-x)$ . This function can be continued analytically. We note that in the numerical calculation the additional condition  $\int \psi(x)\psi(-x)dx = 1$  must be enforced to fix the phase of the non-linearity in the  $\mathcal{PT}$  broken regime.

The substitution of  $|\psi(x)|^2$  by  $\psi(x)\psi(-x)$  allowing analytic continuation of the square modulus of  $\mathcal{PT}$ -symmetric eigenstates shows that the number of solutions does not change at the branch point  $\gamma_{\text{bp}}$ . Yet, at  $\gamma < \gamma_{\text{cr}} < \gamma_{\text{bp}}$  the number of states seem to be less. In fact, the pair of complex eigenvalues bifurcating from the ground state at  $\gamma_{\text{cr}}$  seem to come about without real-valued precursors. However, employing a specific analytic extension of the Gross-Pitaevskii equation these precursors and their corresponding eigenstates can indeed be found.



**Figure 3.** Real parts  $\text{Re}(\kappa)$  (a) and imaginary parts  $\text{Im}(\kappa)$  (b) of the spectrum after the analytical extension for  $g = 1$ . The red solid and green dashed lines represent the  $\mathcal{PT}$ -symmetric and  $\mathcal{PT}$ -broken states already known without any analytic continuation. The new states only visible with the full analytic continuation are marked by blue dotted lines. With these continued states the number of states does no longer change with  $\gamma$ .

## 2.2. Analytic continuation to $\gamma < \gamma_{\text{cr}}$ of the states breaking the $\mathcal{PT}$ -symmetry

The procedure [28] is to express all complex quantities by two real-valued functions, either amplitude and phase, or real and imaginary parts. The non-analytic Gross-Pitaevskii equation then decomposes into two coupled real-valued differential equations which can be continued analytically.

We write for the wave function, the double well potential and the eigenvalue  $\psi = \psi_r + i\psi_i$ ,  $V = V_r + iV_i$  and  $\kappa = \kappa_r + i\kappa_i$ , respectively. Inserting into the Gross-Pitaevskii equation and sorting the real and imaginary contributions leads to

$$-\psi_r'' + V_r\psi_r - V_i\psi_i - g(\psi_r^2 + \psi_i^2)\psi_r = \kappa_r\psi_r - \kappa_i\psi_i, \quad (2a)$$

$$-\psi_i'' + V_r\psi_i + V_i\psi_r - g(\psi_r^2 + \psi_i^2)\psi_i = \kappa_r\psi_i + \kappa_i\psi_r. \quad (2b)$$

The analytical extension consists in allowing the real and imaginary parts of the wave function and the eigenvalue to become complex again:

$$\psi_r = \psi_{rr} + i\psi_{ri}, \quad \psi_i = \psi_{ir} + i\psi_{ii}, \quad (3a)$$

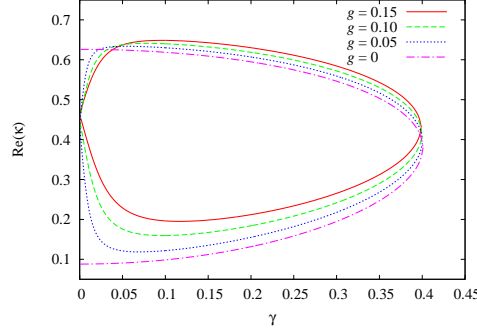
$$\kappa_r = \kappa_{rr} + i\kappa_{ri}, \quad \kappa_i = \kappa_{ir} + i\kappa_{ii}. \quad (3b)$$

When plotting the eigenvalues we recombine the 4 real quantities  $\kappa_{rr}, \kappa_{ri}, \kappa_{ir}, \kappa_{ii}$  into the real and imaginary parts of  $\kappa$ , i. e.

$$\kappa = (\kappa_{rr} - \kappa_{ii}) + i(\kappa_{ri} + \kappa_{ir}) = \text{Re}(\kappa) + i\text{Im}(\kappa). \quad (4)$$

For the value of the non-linearity parameter  $g = 1$  the results for  $\text{Re}(\kappa)$  and  $\text{Im}(\kappa)$  are shown in figure 3. For  $\gamma \leq \gamma_{\text{cr}}$  there are indeed two new branches of real-valued  $\kappa$ . They merge at the triple point and give rise to the complex conjugate pair of eigenvalues. Moreover, these two solutions are present even at  $\gamma = 0$ , where they refer to two degenerate eigenvalues.

It is remarkable that the spectra of the two additional states obtained by the analytic continuation become very similar to the spectra of the original  $\mathcal{PT}$ -symmetric solutions when  $g$  is decreased. This is demonstrated in figure 4, where the spectra shown should be compared with those in figure 1. There is, however, one difference: the



**Figure 4.** Eigenvalue spectra for  $g = 0.15$  (red solid lines),  $g = 0.10$  (green dashed lines), and  $g = 0.05$  (blue dotted lines). For comparison the spectrum without the non-linear term ( $g = 0$ ) is also displayed .

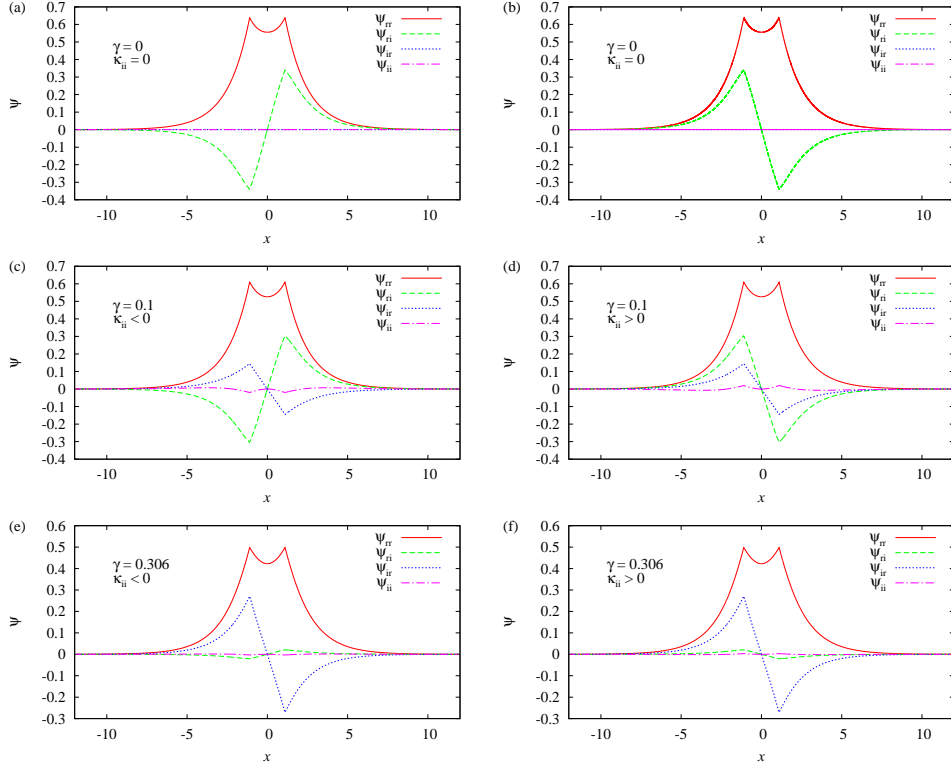
two additional solutions are degenerate at  $\gamma = 0$  for any  $g \neq 0$ , while the eigenvalues of the two original  $\mathcal{PT}$ -symmetric states always have different values at this point.

For  $g > 0$  the wave functions of the analytically continued states shown in figure 4 differ of course from those of the original real eigenvalues where only  $\kappa_{\text{tr}}$  features. It is of interest to have a look at these wave functions. They are illustrated for  $g = 1$  in figure 5 for a few values of  $0 \leq \gamma \leq \gamma_{\text{cr}}$ . Note that the functions are eigenfunctions of  $\mathcal{PT}$  throughout in that their real parts are symmetric and their imaginary parts antisymmetric. Note further that there is a non-vanishing imaginary part  $\psi_{\text{ri}}$  due to the analytic continuation even at  $\gamma = 0$ . At this point the two wave functions are associated with the same energy value as  $\kappa_{\text{ii}} = 0$ , i.e. there is a genuine degeneracy. (This is in contrast to the wave functions of the other two energies appearing for  $\gamma = 0$  (see figure 3)). The imaginary part  $\psi_{\text{ri}}$  tends to zero when  $\gamma_{\text{cr}}$  is approached while another imaginary part  $\psi_{\text{ir}}$  – not due to the analytical continuation – emerges. Eventually, that is at  $\gamma_{\text{cr}}$ , the two wave functions become equal to themselves and to the third (original) state with which they coalesce at the triple point. From this point onwards the states are found without the complex extension, in fact the parts invoked from the extension vanish. Note also, that the additional real part  $\psi_{\text{ii}}$  switches on and off in the interval  $[0, \gamma_{\text{cr}}]$  similar to the behaviour of  $\kappa_{\text{ii}}$ .

### 2.3. Exceptional point behaviour

As pointed out above the three eigenfunctions are identical at the triple point. Therefore the question arises whether or not the triple point is a third-order exceptional point (EP3). The way to test the exceptional point behaviour numerically is to extend one parameter into the complex plane, encircle the point of coalescence and check the interchange of the eigenvalues and states. Figure 6 shows the result for a non-linearity of  $g = 1$  when the triple point at  $\gamma_{\text{cr}} \approx 0.308$  is encircled in the complex extended  $\gamma$  plane. One clearly sees that only the two complex eigenvalues interchange while the ground state remains unaffected. The same holds for the corresponding eigenfunctions. It thus appears that we deal with an EP2 corresponding to the merging of the two new eigenvalues.

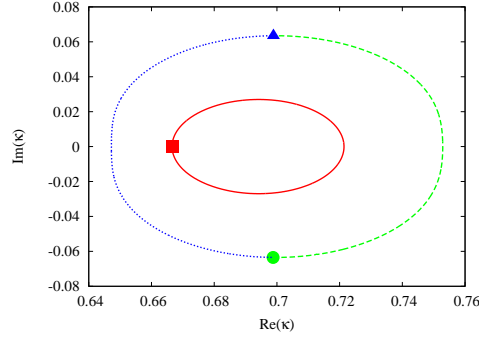
However, when we allow an arbitrarily small asymmetry for the real part of the double well potential by considering  $V_{\text{asym}} = A[\delta(x - a/2) - \delta(x + a/2)]$ , and encircle



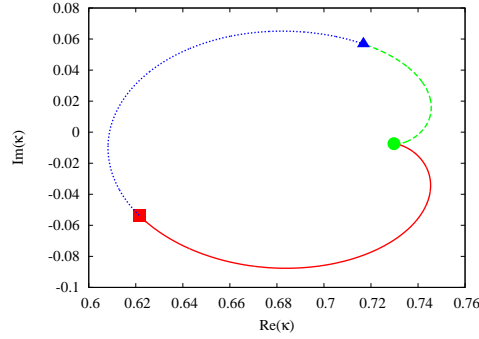
**Figure 5.** Wave functions of the analytically extended solutions for a few values of  $\gamma$  and  $g = 1$ . The right (left) column refers to the lower (upper) branch of the eigenvalues.

now the branch point around  $A = 0$  in the complex extended  $A$  plane, figure 7 clearly points to an EP3. We stress that any other perturbation would produce the same result: the apparent EP2 which does not involve the third state becomes a true EP3 nesting together all three states as soon as the problem becomes disturbed, no matter of what nature and how small the disturbance may be. This behaviour is similar to that of a study of a simple matrix model by Demange and Graefe [29] where three coalescing eigenvectors can exhibit both the behaviour of second- or third-order exceptional point depending on the choice of parameters. The same behaviour has also been found in the investigation of pitchfork bifurcations occurring in Bose-Einstein condensates with dipolar interaction [30].

To summarise: the numerical solution of the  $\mathcal{PT}$ -symmetric Gross-Pitaevskii equation (1) brings about spectral properties calling for deeper insight of the physical system. In the following section the precise nature of these spectral singularities will be clarified in analytic terms by means of a matrix model simulating the properties of the non-linear physical system.



**Figure 6.** Exceptional point behaviour: circle with complex  $\gamma = 0.308 + 0.04e^{i\varphi}$ ,  $\varphi = 0 \dots 2\pi$ , for  $g = 1$ , where  $\gamma \approx 0.308$  is the triple point. For  $\varphi = 0$  we have  $\gamma = 0.312$  and the three states exist without complex continuation. The red square marks the ground state with real  $\kappa$ . Its trace on the parameter space circle is represented by the red solid line. The two states with positive and negative imaginary parts of the eigenvalues  $\kappa$  are represented by the blue triangle (blue dotted line for  $\varphi \neq 0$ ) and the green circle (green dashed line), respectively.



**Figure 7.** Exceptional point behaviour: circle with complex  $A = 0.04e^{i\varphi}$ ,  $\varphi = 0 \dots 2\pi$ , for  $g = 1$  and  $\gamma = 0.308$ . The triple point is located exactly at  $A = 0$ . The symbols and lines have the same meaning as in Fig. 6. The circle in the asymmetry parameter clearly reveals the nature of an EP3.

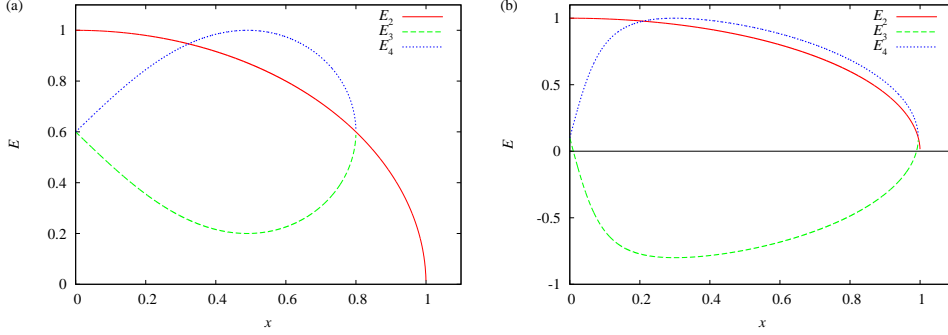
### 3. The Matrix Model

Here we present a simple low-dimensional matrix model that simulates the parameter dependence of the spectrum of the numerical solutions of the Gross-Pitaevskii equation as discussed in the previous section. In particular, the various types of the EPs as found above are of crucial importance: the simulation must reflect the particular types of these spectral singularities.

#### 3.1. General properties

Our starting point is a two-mode model for a  $\mathcal{PT}$ -symmetric Bose-Einstein condensate [6] that mimics well that of the full Gross-Pitaevskii equation. In the two-mode





**Figure 8.** Eigenvalues  $E_2$ ,  $E_3$  and  $E_4$  from Eq. (6) as a function of the non-Hermiticity parameter  $x$  for different strengths of the non-linearity, viz.  $c = 0.6$  (a),  $c = 0.1$  (b). Note the similarity with the spectra in figure 4.

approximation the  $\mathcal{PT}$ -symmetric stationary GP-equation assumes the form

$$\begin{pmatrix} -i\gamma + g|\phi_1|^2 & 1 \\ 1 & +i\gamma + g|\phi_2|^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (5)$$

The off-diagonal element in the Hamiltonian describes the tunnelling between the two modes, and has been normalised to unity. The analytic extension of the GP-equation in the spirit of Ref. [28] then yields the eigenvalues [6]

$$\begin{aligned} E_1 &= -\sqrt{1-x^2}, \\ E_2 &= +\sqrt{1-x^2}, \\ E_3 &= c - x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}}, \\ E_4 &= c + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}}. \end{aligned} \quad (6)$$

Here  $c = g/2$ , and we use the symbol  $x$  instead of  $\gamma$ . By normalising the off-diagonal element to unity we have reduced the relevant interval of  $x$  between 0 and 1. In figure 8 such spectra are illustrated for some values of  $c$ . It is striking how well these eigenvalues quantitatively describe the behaviour of those found numerically from the GP-equation. This perfect simulation includes in particular the spectral singularities and also allows to study the precise behaviour of the non-uniform limit  $g \rightarrow 0$  appearing here as  $c \rightarrow 0$ .

At the critical point  $x = \sqrt{1-c^2}$  the three eigenvalues  $E_2, E_3$  and  $E_4$  all assume the same value  $c$ . Beyond this point  $E_3$  and  $E_4$  become complex; in fact,  $E_3$  and  $E_4$  have a square root singularity that appears to have some characteristics of a usual EP. The precise nature, especially its interplay with  $E_2$ , cannot be deduced from the spectrum alone. It is the interaction giving rise to particular wave functions that determines the precise nature of a spectral singularity. As we are interested in the intersection point of the three levels  $E_2, E_3$  and  $E_4$  we restrict ourselves to a three-dimensional matrix model and leave out the level  $E_1$  altogether as it seems immaterial to our specific interest here.

We seek a three-dimensional matrix which produces the spectrum  $\{E_2, E_3, E_4\}$ , but in such a way that all levels are fully interacting. This is in contrast to the matrix

given in [6] where an interaction between the two branches  $E_1, E_2$  and  $E_3, E_4$  is not taken into account. In other words, we seek a similarity transformation  $s$  such that

$$h = s \cdot j \cdot s^{-1} \quad (7)$$

where  $j$  is diagonal with the levels  $\{E_2, E_3, E_4\}$  in its diagonal and  $s$  invokes the interaction between the levels. For our purpose the matrix  $s$  must bear the singularities of  $j$ . Moreover, a particular rank drop at the critical point  $x = \sqrt{1 - c^2}$  determines as to whether the spectrum is simply degenerate, or whether it has a coalescence of two or of three levels, the former being a usual EP2 and the latter an EP3. Note that  $s^{-1}$  does not exist if  $s$  does not have full rank whereas  $h$  is still well defined but may no longer be diagonalisable; this is the signature of an EP. If  $\text{rank}(s)=1$  at the critical point, the Jordan normal form of  $h$  is expected to be

$$J[h] = \begin{pmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix}, \quad (8)$$

being the signature of an EP3: three levels are coalescing.

A judicious choice fulfilling these requirements is given by

$$s = \begin{pmatrix} 1 & \sqrt{1-x^2} & \sqrt{1-x^2} \\ 1 & c - x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}} & c + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}} \\ 1 & (\sqrt{c} - x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^2 & (\sqrt{c} + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^2 \end{pmatrix}. \quad (9)$$

It can be checked that  $h$  yields Eq. (8) at  $x = \sqrt{1 - c^2}$ . Moreover, for  $x \neq \sqrt{1 - c^2}$  one sees that encircling the critical point in the complex plane swaps the levels  $E_3$  and  $E_4$  including their (unnormalised) eigenfunctions obtained from Eq. (9)

$$\begin{pmatrix} \sqrt{1-x^2} \\ c - x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}} \\ (\sqrt{c} - x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{1-x^2} \\ c + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}} \\ (\sqrt{c} + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^2 \end{pmatrix}$$

being just the characteristic of an EP2. In fact, the square root  $\sqrt{1 - x^2 - c^2}$  simply changes sign when a circle is traced out in the complex  $x$ -plane around the point  $\sqrt{1 - c^2}$  thereby interchanging  $E_3$  and  $E_4$  and their corresponding eigenvectors. Note, however, that *at the critical point* all three eigenvectors also become equal (up to a constant) thus indicating the special nature of this point.

We emphasise that the clear signature of an EP3 as presented above is only available when analytic expressions are at hand. To obtain this result directly by numerical means is virtually impossible. One rather has to resort to the means as presented in the previous section, where the existence of the EP3 is established by invoking an asymmetry and then use closed contours in a suitable complex parameter plane. To elucidate this pattern we discuss in the appendix a three-dimensional matrix demonstrating the underlying mechanism; as our matrix  $h$  is unsuitable for demonstration due to its complicated structure we use a much simplified matrix.

### 3.2. Behaviour for $c \rightarrow 0$

All these statements hold for  $c > 0$  but care must be taken when  $c \rightarrow 0$ . Recall that the GP-equation has a non-uniform limit for  $g \rightarrow 0$  as is discussed in Section 2. The

same holds for the matrix model when  $c \rightarrow 0$ . As the limiting behaviour is subtle and difficult to obtain numerically, we here give some results obtained in the matrix model. Note in particular how the triple point approaches the branch point where the  $\mathcal{PT}$ -eigenfunctions meet when  $c \rightarrow 0$  for the matrix model (figure 8) and  $g \rightarrow 0$  for the numerical solution (figure 4).

The eigenvalues  $E_3$  and  $E_4$  have, for  $c > 0$ , the value  $c$  at  $x = 0$ . Yet these eigenvalues are  $\pm 1$  for  $c = 0$  and  $x = 0$ . Similarly, the derivative tends to infinity at  $x = 0$  for  $c \rightarrow 0$ , as can be checked analytically, and is visualized also in figure 8. But it is zero when  $c = 0$  is taken from the outset. In other words,  $E_4$  is getting nearer and nearer to  $E_2$ , but at  $x = 0$  in a non-continuous fashion ( $E_3$  approaches  $-E_4$  for  $c \rightarrow 0$ ). Note that this behaviour nicely reflects the behaviour of the numerical solutions of the full GP-equation (see figure 4).

At  $x = 1$  it is the eigenfunctions that show the typical non-uniform behaviour. Note that the eigenvectors as listed in Eq. (9) cannot be used in the limits  $x \rightarrow 1$  and  $c \rightarrow 0$ , irrespective of the order of the limits taken, since  $s^{-1}$  does not exist. Rather, the full matrix  $h$  must be considered. If  $c = 0$ , corresponding to switching off the non-linearity in the GP-equation, the eigenvectors of the two positive solutions ( $E_2 = E_4 = \sqrt{1 - x^2}$ ) are given by

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

whereas, when the limits are commuted ( $x = 1$  first and then  $c \rightarrow 0$ ) the corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ 1 + i \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 - i \\ 1 \end{pmatrix}.$$

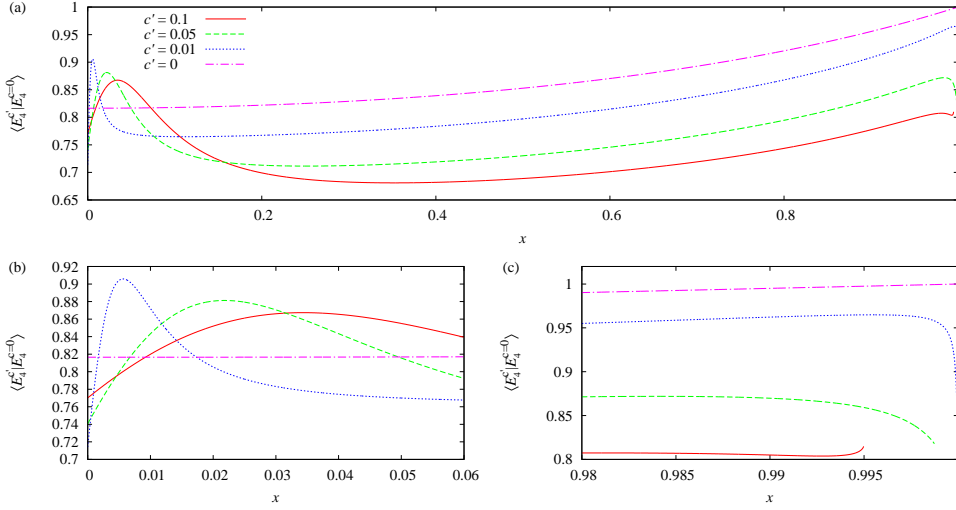
Recall that  $E_3$  and  $E_4$  are complex when  $x$  is to the right of the critical point, that is if  $x > \sqrt{1 - c^2}$ , a condition met for the order of the limits taken. This is reflected here in the complex eigenvectors, even though the eigenvalues are zero in either limit.

A further point of interest is the behaviour of the scalar product taken from the eigenvectors referring to the plain solution for  $c = 0$  and the eigenvector referring to  $c > 0$ . Of interest is the limit  $c \rightarrow 0$  of the scalar product. For the eigenvalue  $E_4 = c + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}}$  we obtain for the normalised eigenvectors the scalar product

$$\langle E_4^{c>0} | E_4^{c=0} \rangle = \frac{c + \sqrt{1 - x^2} + x\sqrt{\frac{1-c^2-x^2}{c^2+x^2}}}{\sqrt{2}\sqrt{1 - x^2 + (\sqrt{c} + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^4 + (c + x\sqrt{\frac{1-x^2-c^2}{x^2+c^2}})^2}}. \quad (10)$$

We simply list some major facts as they can be easily verified from Eq. (10). The scalar product behaves in its limit  $c \rightarrow 0$  non-uniformly at both ends of the  $x$ -interval. At  $x = 0$  Eq. (10) assumes the value  $(1 + c)/\sqrt{2 + 4c^2}$  whereas the value  $\sqrt{2/3}$  is assumed when  $c = 0$  is set first. At the right end of the  $x$ -interval the value is unity when  $c = 0$  while the value is  $\sqrt{(1/2 + i)(3 - i)}/5$  when  $x = 1$  is set before  $c \rightarrow 0$ . We mention that the intricacy of the expansion in powers of  $c$  probably hints to the difficulty in approaching the limit  $c \rightarrow 0$  numerically using the GP-equation directly. In fact the expansion reads

$$\frac{\sqrt{1 - x^2}\sqrt{2}}{\sqrt{3 - 4x^2 + x^4}} - \sqrt{c}\frac{2\sqrt{2}(x^2 - 1)}{(x^2 - 3)\sqrt{3 - 4x^2 + x^4}} + \dots$$



**Figure 9.** Scalar product of the eigenvector for  $c = 0$  with the eigenvector for  $c' \neq 0$  as function of  $x$  in the range  $0 \leq x \leq 1$  (a). The values displayed refer to  $c' = 0.1$  (red solid line),  $c' = 0.05$  (green dashed line),  $c' = 0.01$  (blue dotted line) and  $c' \rightarrow 0$  (purple dashed-dotted line). For better illustration the  $x$ -axis is stretched at  $x = 0$  (b) and at  $x = 1$  (c). All lines are purely real only for  $x < \sqrt{1 - c^2}$ ; it explains the ending for  $x < 1$  as only pure real values are displayed.

where the  $x$ -dependent coefficient of  $\sqrt{c}$  vanishes for  $x = 1$ ; it means that the  $\sqrt{c}$ -behaviour will show for  $x < 1$  but not for  $x = 1$ . In figure 9 the scalar product is illustrated for a few values of  $c$ . The non-uniform behaviour at both ends of the  $x$ -interval is clearly visible. We note that the scalar product assumes the value  $\sqrt{2/3}$  at the critical point, independent of  $c$ .

#### 4. Summary and Conclusions

A number of novel aspects originating from the non-linear term  $g|\psi|^2$  in the interaction term of the Schrödinger equation are communicated in the present paper. There is the occurrence of two new states in the low energy spectrum. Naively, they are seen for  $g > 0$  only when they emerge as complex solutions from the ground state for  $\gamma \geq \gamma_{\text{cr}}$  thereby breaking the  $\mathcal{PT}$ -symmetry of the system. An appropriate complex extension of the GP-equation reveals that two real additional solutions do exist for  $\gamma < \gamma_{\text{cr}}$  which are in fact eigenfunctions of  $\mathcal{PT}$ . A further novel result is the triple point at  $\gamma = \gamma_{\text{cr}}$ , in particular the singular spectral behaviour. With a superficial look the singularity appears as a “common” EP2 where the two new solutions turn complex while the ground state from which they emerge seems to be unaffected. However, a more thorough analysis reveals that the three states are nested in an EP3; in fact, for any generic perturbation this result prevails.

While these results are obtained by careful numerical analysis the point  $g = 0$  being a singular point of the non-linear equation seems to be inaccessible by numerical means. In other words, the limit  $g \rightarrow 0$  leading to the simpler linear equation is subtle, there is a clear non-uniform behaviour for the spectrum and eigenfunctions. To study

this aspect in analytic terms a matrix model simulating quantitatively the spectrum of the GP-equation is presented. While the model elucidates the limit very clearly it also sheds light on the specific behaviour of the triple point: the distinction between the EP2 and EP3 behaviour depending on the specific approach of the singular point becomes evident in analytic terms. The matrix model is designed to be as simple and instructive as possible since our interest is focussed upon the explanation of the spectra. An exact quantitative agreement with the numerical results is not intended as it will complicate the analytic discussion.

We believe that these findings are expected to have an impact in experimental work in that specific effects can be detected related to the structure of the model studied. It is well known that EPs have a definite physical meaning [31] in a great variety of physical systems. The presence of the triple point has already been established to lead to an instability in the dynamical behaviour [9]. It remains a challenge to detect the signature of the EP3 in a BEC. Whether or not even the two additional states have physical significance is, at this stage, an open question.

### Acknowledgments

Work on this topic began at the European Centre for Theoretical Studies in Nuclear Physics and Related Areas (ECT\*), Trento, Italy. We thank the Centre and its then director Achim Richter for their hospitality and generous support. WDH and GW also gratefully acknowledge the support from the National Institute for Theoretical Physics (NITheP), Western Cape, South Africa. GW expresses his gratitude to the Department of Physics of the University of Stellenbosch for kind hospitality while this manuscript was prepared.

### Appendix

The matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ 2+c+x & -1 & -c-x \\ -1 & 0 & 2 \end{pmatrix}$$

has the spectrum

$$\begin{aligned} E_1 &= 1 \\ E_2 &= 1 + \sqrt{1-c-x} \\ E_3 &= 1 - \sqrt{1-c-x}. \end{aligned}$$

Clearly, and from the structure of the associated eigenfunctions, there is an EP2 at  $x = 1 - c$ . However, when  $x = 1 - c$  is chosen from the outset the Jordan normal form reads

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which clearly indicates an EP3. How to find this by numerical means?

We use a perturbation  $\epsilon$

$$\begin{pmatrix} 2+\epsilon & -1 & 0 \\ 2+c+x & -1 & -c-x \\ -1 & 0 & 2 \end{pmatrix}.$$

The expansions of the eigenvalues read

$$\begin{aligned} E_1 &= 1 + \frac{2}{1-c-x}\epsilon + O(\epsilon^2) \\ E_2 &= 1 + \sqrt{1-c-x} + \left( \frac{1+c+x}{2(1-c-x)} + \frac{1}{2\sqrt{1-c-x}} \right) \epsilon + O(\epsilon^2) \\ E_3 &= 1 - \sqrt{1-c-x} + \left( \frac{1+c+x}{2(1-c-x)} - \frac{1}{2\sqrt{1-c-x}} \right) \epsilon + O(\epsilon^2). \end{aligned}$$

Obviously, the expansion breaks down at the critical point  $x = 1 - c$ . When  $x$  is set equal to  $1 - c$  in the perturbed matrix, we now see clearly how the eigenvalues sprout from the EP3 in the expected manner. The expansions read now

$$\begin{aligned} E_1 &= 1 - 2^{1/3} \exp(2i\pi/3) \epsilon^{1/3} + O(\epsilon^{2/3}) \\ E_2 &= 1 - 2^{1/3} \epsilon^{1/3} + O(\epsilon^{2/3}) \\ E_3 &= 1 - 2^{1/3} \exp(-2i\pi/3) \epsilon^{1/3} + O(\epsilon^{2/3}). \end{aligned}$$

This pattern is basically unchanged for any generic perturbation. It clearly demonstrates the principle underlying in the numerical manifestation of the EP3.

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